Applications of Group Theory

Let us now look at what are the applications of group theory in mathematics.

In Mathematics and abstract algebra, group theory studies the algebraic structures that are called groups. The concept of the group is a center to abstract algebra. The other well-known algebraic structures like the rings, fields and vector spaces are all seen as the groups that are endowed with the additional operations and axioms. Groups recur throughout when it comes to mathematics, and the methods of group theory have influenced several parts of algebra. The linear algebraic groups and the Lie groups are the two branches of group theory that have experienced advances and are the subject areas in their own ways.

Several physical systems like the crystals and the hydrogen atom can be modeled by the symmetry groups. Hence the group theory and the closely related theory called the representation theory to have several important applications in the fields of physics, material science, and chemistry. The group theory is also the center of public-key cryptography.

Group Theory

Let us look at some of the group theory Lemmas.

Lemma 1: Let G be a group. Prove that the element $e \in$

G is unique. Also, prove that each of the elements $x \in$

G consists of a unique inverse which is denoted by x^{-1}

Solution:
Consider e and e' to be the identities.

According to the definition, you get $e' = e * e' = e$.

Similarly, consider y and y' to be the inverses of x.

Then, you would get

y = y * e

= y * (x * y')

= (y * x) * y'

= e * y'

= y'

Lemma 2: Consider x, y ∈ G having the inverses x^{-1} and y^{-1} respectively. Determine the inverse of xy.

Solution: The inverse of the product of x and y is given as follows:

$x * y = x^{-1} * y^{-1}$

You have $(x * y) * (x^{-1} * y^{-1}) = x (y * y^{-1}) x^{-1} = xex^{-1} = e$

Similarly,

(x−1 * y-1) * (x * y) = e

Therefore, (xy)−1

= x−1

Y−1

Lemma 3: If G is a group that has a and b as its elements, such that a, b ∈ G, then $(a \times b)^{-1} = a^{-1} \times b^{-1}$

Proof:

To prove: $(a \times b) \times b^{-1} \times a^{-1} = I$, where I is the identity element of G.

Consider the L.H.S of the above equation, we have,

L.H.S = $(a \times b) \times b^{-1} \times b^{-1}$

=> $a \times (b \times b^{-1}) \times b^{-1}$

=> $a \times I \times a^{-1}$ (by associative axiom)

=> $(a \times I) \times a^{-1}$ (by identity axiom)

= $a \times a^{-1}$ (by identity axiom)

= I (by identity axiom)

= R.H.S

Hence, proved.

Lemma 4: If in a group G, 'x', 'y' and 'z' are three elements such that x × y = z × y, then x = z.

Proof: Let us assume that x × y = z × y. (i)

Since ‘y’ is an element of group G, this implies there exist some ‘a’ in G with identity element I, such that;

y × a = I (ii)

On multiplying both sides of (i) by ‘a’ we get,

x × y × a = z × y × a

x × (y × a) = z × (y × a) (by associativity)

From eq.(ii);

a × I = c × I [using (ii)]

a = c (by identity axiom)

This is also known as cancellation law.

Hence, proved.